

DYNAMICS OF COSMOLOGICAL PHASE TRANSITIONS¹

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Abstract

The dynamics of typical phase transitions is studied out of equilibrium in weakly coupled inflaton-type scalar field theories in Minkowski space. The shortcomings of the effective potential and equilibrium descriptions are pointed out. A case of a rapid supercooling from $T_i > T_c$ to $T_f \ll T_c$ is considered. The equations of motion up to one-loop for the order parameter are obtained and integrated for the case of “slow rollover initial conditions”. It is shown that the instabilities responsible for the process of phase separation introduce dramatic corrections to the evolution. Domain formation and growth (spinodal decomposition) is studied in a non-perturbative self-consistent approximation. For very weakly coupled theories domains grow for a long time, their final size is several times the zero temperature correlation length. For strongly coupled theories the final size of the domains is comparable to the zero temperature correlation length and the transition proceeds faster. We also obtain the evolution equations for the order parameter and the fluctuations to one-loop order and in a non-perturbative Hartree approximation in spatially flat FRW cosmologies. The renormalization, and leading behavior of the high temperature limit are analyzed.

1 Introduction and Motivation

Inflationary cosmological models provide very appealing scenarios to describe the early evolution of our universe[1]. Since the original model proposed by Guth[2], several alternative scenarios have been proposed to overcome some of the difficulties with the original proposal.

Among them, the new inflationary model[3, 4, 5, 6] is perhaps one of the most attractive. The essential ingredient in the new inflationary model is a scalar field (the “inflaton”) that undergoes a second order phase transition from a high temperature symmetric phase to a low temperature broken symmetry phase. The expectation value (or thermal average) of the scalar field ϕ serves as the order parameter. Initially at high temperatures, the scalar field

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is assumed to be in thermal equilibrium and $\phi \approx 0$. The usual field-theoretic tool to study the phase transition is the effective potential[7, 8, 9].

At high temperatures, the global minimum of the effective potential is at $\phi = 0$, whereas at low temperatures there are two degenerate minima.

The conjectured behavior of the phase transition in the new inflationary model is the following: as the universe cools down, the expectation value of the scalar field remains close to zero until the temperature becomes smaller than the critical temperature at which the effective potential develops degenerate minima away from the origin. When this happens, the scalar field begins to “roll down the potential hill”. In the new inflationary scenario, the effective potential below the critical temperature is extremely flat near the maximum, and the scalar field remains near the origin, i.e. the false vacuum, for a very long time and eventually rolls down the hill very slowly, hence the name “slow rollover”. During this stage, the energy density of the universe is dominated by the constant energy density of the false vacuum $V_{eff}(\phi = 0)$, and the universe evolves rapidly into a de Sitter space (see for example the reviews by Kolb and Turner[10], Linde[11] and Brandenberger[12]). Perhaps the most remarkable consequence of the new inflationary scenario and the slow rollover transition is that they provide a calculational framework for the prediction of density fluctuations[13]. The coupling constant in the typical zero temperature potentials must be fine tuned to a very small value to reproduce the observed limits on density fluctuations[10, 11].

The slow rollover scenario is based on the *static* effective potential. Its usefulness in a time dependent situation has been questioned by Mazenko, Unruh and Wald[14] who argued that the *dynamics* of the cooling down process is very similar to the process of phase separation in statistical mechanics, and that the system will form domains within which the scalar field will relax to the values at the minima of the potential very quickly. It is now accepted that this picture may be correct for very strongly coupled theories but not for weakly coupled scenarios as required in inflation.

Guth and Pi[15], studied the effects of quantum fluctuations on the time evolution below the critical temperature by treating the potential near the origin as an *inverted harmonic oscillator*. They recognized that the instabilities associated with these upside-down oscillators lead to an exponential growth of the quantum fluctuations at long times and to a classical description of the probability distribution function. Guth and Pi also recognized that the *static* effective potential is not appropriate to describe the dynamics, that must be treated as a time dependent process.

Subsequently, Weinberg and Wu[16], have studied the effective potential, particularly in the situation when the tree level potential allows for broken symmetry ground states. In this case, the effective potential becomes complex. These authors identified the imaginary part of the effective potential with the decay rate of a particular initial state.

In statistical mechanics, the imaginary part is a consequence of a sequence of thermodynamically unstable states.

The effective potential offers no information on the *dynamics* of the process of the phase transition. As mentioned above it becomes complex in a region in field space corresponding to thermodynamically unstable states.

Furthermore, in an expanding gravitational background the notion of a static effective potential is at best an approximate one. To understand whether one may treat the problem in the approximation of local thermodynamic equilibrium, the time scales involved must be understood carefully. In a typical FRW cosmology the important time scale is determined by the Hubble expansion factor $H(t) = \dot{a}(t)/a(t)$ whereas the equilibration processes are determined by the typical collisional relaxation rates $\Gamma(T) \approx \lambda^2 T$ in typical scalar theories. Local thermodynamic equilibrium will prevail if $\Gamma(T) \gg H(t)$. In de Sitter evolution, after a typical phase transition at $T \approx T_c \approx 10^{14}$ GeV, $H \approx 10^{-5} T_c$ [11] and local thermodynamic equilibrium *will not* prevail for weakly coupled inflaton theories in which $\lambda \approx 10^{-12} - 10^{-14}$ [11, 13, 20].

Thus in these scenarios, weakly coupled scalar field theories *cannot be assumed* to be in local thermal equilibrium. Typical expansion rates are much larger than typical equilibration rates and the phase transition will occur very rapidly. The long-wavelength fluctuations will be strongly out of equilibrium as they typically have very slow dynamics (see [17] and references therein). The phase transition *must* be studied away from equilibrium.

A complete discussion of these issues and the complex effective potential may be found in the articles by Boyanovsky and de Vega [18] and Boyanovsky et. al. [17, 19].

2 Non-Equilibrium time evolution

Let us consider the situation in which for time $t < 0$ the system is in *equilibrium* at an initial temperature $T_i > T_c$ where T_c is the critical temperature. At time $t = 0$ the system is rapidly “quenched” to a final temperature below the critical temperature $T_f < T_c$ and evolves thereafter out of equilibrium.

What we have in mind in this situation, is a cosmological scenario with a period of rapid inflation in which the temperature drops very fast compared to typical relaxation times of the scalar field.

Precisely because of the weak couplings and critical slowing down of long-wavelength fluctuations, we conjecture that an inflationary period at temperatures near the critical temperature, may be described in this “quenched” approximation. Another situation that may be described by this approximation is that of a scalar field again at $T_i > T_c$ suddenly coupled to a “heat bath” at a much lower temperature (below the transition temperature) and evolving out of equilibrium.

The “heat bath” could consist of other fields at a different temperature.

To understand whether this quenched approximation is valid or not will require a deeper understanding of the initial conditions.

Although we are currently studying the case of inflationary cosmologies, we will here concentrate on the dynamics of a supercooled phase transition in Minkowski space.

This situation is modelled by introducing a Hamiltonian with a *time dependent mass*

term

$$H(t) = \int_{\Omega} d^3x \left\{ \frac{1}{2} \Pi^2(x) + \frac{1}{2} (\vec{\nabla} \Phi(x))^2 + \frac{1}{2} m^2(t) \Phi^2(x) + \frac{\lambda}{4!} \Phi^4(x) \right\} \quad (1)$$

$$m^2(t) = m^2 \Theta(-t) + (-\mu^2) \Theta(t) \quad (2)$$

where both m^2 and μ^2 are positive. We assume that for all times $t < 0$ there is thermal equilibrium at temperature T_i , and the system is described by the density matrix

$$\hat{\rho}_i = e^{-\beta_i H_i} \quad (3)$$

$$H_i = H(t < 0) \quad (4)$$

In the Schroedinger picture, the density matrix evolves in time as

$$\hat{\rho}(t) = U(t) \hat{\rho}_i U^{-1}(t) \quad (5)$$

with $U(t)$ the time evolution operator.

An alternative and equally valid interpretation (and the one that we like best) is that the initial condition being considered here is that of a system in equilibrium in the symmetric phase which is then evolved in time with a Hamiltonian that allows for broken symmetry ground states, i.e. the Hamiltonian (1, 2) for $t > 0$.

The expectation value of any operator is thus

$$\langle \mathcal{O} \rangle(t) = \text{Tr} e^{-\beta_i H_i} U^{-1}(t) \mathcal{O} U(t) / \text{Tr} e^{-\beta_i H_i} \quad (6)$$

This expression may be written in a more illuminating form by choosing an arbitrary time $T < 0$ for which $U(T) = \exp[-iTH_i]$ then we may write $\exp[-\beta_i H_i] = \exp[-iH_i(T - i\beta_i - T)] = U(T - i\beta_i, T)$. Inserting in the trace $U^{-1}(T)U(T) = 1$, commuting $U^{-1}(T)$ with $\hat{\rho}_i$ and using the composition property of the evolution operator, we may write (6) as

$$\langle \mathcal{O} \rangle(t) = \text{Tr} U(T - i\beta_i, t) \mathcal{O} U(t, T) / \text{Tr} U(T - i\beta_i, T) \quad (7)$$

The numerator of the above expression has a simple meaning: start at time $T < 0$, evolve to time t , insert the operator \mathcal{O} and evolve backwards in time from t to $T < 0$, and along the negative imaginary axis from T to $T - i\beta_i$. The denominator, just evolves along the negative imaginary axis from T to $T - i\beta_i$. The contour in the numerator may be extended to an arbitrary large positive time T' by inserting $U(t, T')U(T', t) = 1$ to the left of \mathcal{O} in (7), thus becoming

$$\langle \mathcal{O} \rangle(t) = \text{Tr} U(T - i\beta_i, T) U(T, T') U(T', t) \mathcal{O} U(t, T) / \text{Tr} U(T - i\beta_i, T) \quad (8)$$

The numerator now represents the process of evolving from $T < 0$ to t , inserting the operator \mathcal{O} , evolving further to T' , and backwards from T' to T and down the negative imaginary axis to $T - i\beta_i$. Eventually we take $T \rightarrow -\infty$; $T' \rightarrow \infty$. It is straightforward to generalize to real time correlation functions of Heisenberg picture operators.

As usual, the insertion of an operator may be achieved by inserting sources in the time evolution operators, defining the generating functionals and eventually taking functional derivatives with respect to these sources. Notice that we have three evolution operators, from T to T' , from T' , back to T (inverse operator) and from T to $T - i\beta_i$.

Since each of these operators has interactions and we want to generate the diagrammatics of perturbation theory from the generating functionals, we use *three different sources*. A source J^+ for the evolution $T \rightarrow T'$, J^- for the branch $T' \rightarrow T$ and finally J^β for $T \rightarrow T - i\beta_i$. The denominator may be obtained from the numerator by setting $J^+ = J^- = 0$. Finally the generating functional $Z[J^+, J^-, J^\beta] = Tr U(T - i\beta_i, T; J^\beta) U(T, T'; J^-) U(T', T; J^+)$, may be written in term of path integrals as (here we neglect the spatial arguments to avoid cluttering of notation)

$$Z[J^+, J^-, J^\beta] = \int D\Phi D\Phi_1 D\Phi_2 \int \mathcal{D}\Phi^+ \mathcal{D}\Phi^- \mathcal{D}\Phi^\beta e^{i \int_T^{T'} \{ \mathcal{L}[\Phi^+, J^+] - \mathcal{L}[\Phi^-, J^-] \}} \times e^{i \int_T^{T-i\beta_i} \mathcal{L}[\Phi^\beta, J^\beta]} \quad (9)$$

with the boundary conditions $\Phi^+(T) = \Phi^\beta(T - i\beta_i) = \Phi$; $\Phi^+(T') = \Phi^-(T') = \Phi_2$; $\Phi^-(T) = \Phi^\beta(T) = \Phi_1$. As usual the path integrals over the quadratic forms may be done and one obtains the final result for the partition function

$$Z[J^+, J^-, J^\beta] = e^{\left\{ i \int_T^{T'} dt [\mathcal{L}_{int}(-i\delta/\delta J^+) - \mathcal{L}_{int}(i\delta/\delta J^-)] \right\}} e^{\left\{ i \int_T^{T-i\beta_i} dt \mathcal{L}_{int}(-i\delta/\delta J^\beta) \right\}} \times e^{\left\{ \frac{i}{2} \int_c dt_1 \int_c dt_2 J_c(t_1) J_c(t_2) G_c(t_1, t_2) \right\}} \quad (10)$$

Here J_c are the currents defined on the segments of the contour[18] J^\pm , J^β [23] and G_c is the Green's function on the contour (see below), and again the spatial arguments have been suppressed.

In the two contour integrals (on t_1 ; t_2) in (10) there are altogether nine terms, corresponding to the combination of currents in each of the three branches. However, in the limit $T \rightarrow -\infty$, the contributions arising from the terms in which one current is on the (+) or (-) branch and another on the imaginary time segment (from T to $T - i\beta_i$), go to zero when computing correlation functions in which the external legs are at finite *real time*. For these *real time correlation functions* there is no contribution from the J^β terms. These cancel between numerator and denominator, and the information on finite temperature is encoded in the boundary conditions on the Green's functions (see below). Then for the calculation of finite *real time* correlation functions the generating functional simplifies to[24, 25]

$$Z[J^+, J^-] = e^{\left\{ i \int_T^{T'} dt [\mathcal{L}_{int}(-i\delta/\delta J^+) - \mathcal{L}_{int}(i\delta/\delta J^-)] \right\}} \times e^{\left\{ \frac{i}{2} \int_T^{T'} dt_1 \int_T^{T'} dt_2 J_a(t_1) J_b(t_2) G_{ab}(t_1, t_2) \right\}} \quad (11)$$

with $a, b = +, -$.

This formulation in terms of time evolution along a contour in complex time has been used many times in non-equilibrium statistical mechanics. To our knowledge the first to

use this formulation were Schwinger[26] and Keldysh[27] (see also Mills[28]). There are many articles in the literature using these techniques to study time dependent problems, some of the more clear articles are by Jordan[31], Niemi and Semenoff[23], Landsman and van Weert[29], Semenoff and Weiss[30], Kobes and Kowalski[32], Calzetta and Hu[25], Paz[33] and references therein (for more details see[18, 17, 19]).

The Green's functions that enter in the integrals along the contours in (10, 11) are given by (see above references)

$$G^{++}(t_1, t_2) = G^>(t_1, t_2)\Theta(t_1 - t_2) + G^<(t_1, t_2)\Theta(t_2 - t_1) \quad (12)$$

$$G^{--}(t_1, t_2) = G^>(t_1, t_2)\Theta(t_2 - t_1) + G^<(t_1, t_2)\Theta(t_1 - t_2) \quad (13)$$

$$G^{+-}(t_1, t_2) = -G^<(t_1, t_2) \quad (14)$$

$$G^{-+}(t_1, t_2) = -G^>(t_1, t_2) = -G^<(t_2, t_1) \quad (15)$$

$$G^<(T, t_2) = G^>(T - i\beta_i, t_2) \quad (16)$$

As usual $G^<, G^>$ are homogeneous solutions of the quadratic form with appropriate boundary conditions. We will construct them explicitly later. The condition (16) is recognized as the periodicity condition in imaginary time (KMS condition)[34].

To obtain the evolution equations we use the tadpole method[9], and write

$$\Phi^\pm(\vec{x}, t) = \phi(t) + \Psi^\pm(\vec{x}, t) \quad (17)$$

Where, again, the \pm refer to the branches for forward and backward time propagation. The reason for shifting both (\pm) fields by the *same* classical configuration, is that ϕ enters in the time evolution operator as a background c-number variable, and time evolution forward and backwards are now considered in this background.

The evolution equations are obtained with the tadpole method by expanding the Lagrangian around $\phi(t)$ and considering the *linear*, cubic, quartic, and higher order terms in Ψ^\pm as perturbations and requiring that

$$< \Psi^\pm(\vec{x}, t) > = 0.$$

It is a straightforward exercise to see that this is equivalent to extremizing the one-loop effective action in which the determinant (in the logdet) incorporates the boundary condition of equilibrium at time $t < 0$ at the initial temperature.

To one loop we find the equation of motion

$$\frac{d^2\phi(t)}{dt^2} + m^2(t)\phi(t) + \frac{\lambda}{6}\phi^3(t) + \frac{\lambda}{2}\phi(t) \int \frac{d^3k}{(2\pi)^3} (-i)G_k(t, t) = 0 \quad (18)$$

with $G_k(t, t) = G_k^<(t, t) = G_k^>(t, t)$ is the spatial Fourier transform of the equal-time Green's function.

Notice that

$$(-iG_k(t, t)) = < \Psi_k^+(t) \Psi_{-k}^+(t) >$$

is a *positive definite quantity* (because the field Ψ is real) and as we argued before (and will be seen explicitly shortly) this Green's function grows in time because of the instabilities associated with the phase transition and domain growth[15, 16].

These Green's functions are constructed out of the homogeneous solutions to the operator of quadratic fluctuations

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + M^2(t) \right] \mathcal{U}_k^\pm = 0 \quad (19)$$

$$M^2(t) = (m^2 + \frac{\lambda}{2}\phi_i^2)\Theta(-t) + (-\mu^2 + \frac{\lambda}{2}\phi^2(t))\Theta(t) \quad (20)$$

The boundary conditions on the homogeneous solutions are

$$\mathcal{U}_k^\pm(t < 0) = e^{\mp i\omega_<(k)t} \quad (21)$$

$$\omega_<(k) = \left[\vec{k}^2 + m^2 + \frac{\lambda}{2}\phi_i^2 \right]^{\frac{1}{2}} \quad (22)$$

where ϕ_i is the value of the classical field at time $t < 0$ and is the initial boundary condition on the equation of motion. Truly speaking, starting in a fully symmetric phase will force $\phi_i = 0$, and the time evolution will maintain this value. Therefore we admit a small explicit symmetry breaking field in the initial density matrix to allow for a small ϕ_i . The introduction of this initial condition seems artificial since we are studying the situation of cooling down from the symmetric phase. However, we recognize that the phase transition from the symmetric phase occurs via formation of domains (in the case of a discrete symmetry) inside which the order parameter acquires non-zero values. The domains will have the same probability for either value of the field and the volume average of the field will remain zero. These domains will grow in time, this is the phenomenon of phase separation and spinodal decomposition familiar in condensed matter physics. Our evolution equations presumably will apply to the coarse grained average of the scalar field inside each of these domains. This average will only depend on time. Thus, we interpret ϕ_i as corresponding to the coarse grained average of the field in each of these domains. The question of initial conditions on the scalar field is also present (but usually overlooked) in the slow-rollover scenarios but as we will see later, it plays a fundamental role in the description of the evolution.

The identification of the initial value ϕ_i with the average of the field in each domain is certainly a plausibility argument to justify an initially small asymmetry in the scalar field which is necessary for the further evolution of the field, and is consistent with the usual assumption within the slow rollover scenario.

The boundary conditions on the mode functions $\mathcal{U}_k^\pm(t)$ correspond to “vacuum” boundary conditions of positive and negative frequency modes (particles and antiparticles) for $t < 0$.

Finite temperature enters through the periodicity conditions (16) and the Green's functions are

$$G_k^>(t, t') = \frac{i}{2\omega_<(k)} \frac{1}{1 - e^{-\beta_i\omega_<(k)}} \left[\mathcal{U}_k^+(t)\mathcal{U}_k^-(t') + e^{-\beta_i\omega_<(k)}\mathcal{U}_k^-(t)\mathcal{U}_k^+(t') \right] \quad (23)$$

$$G_k^<(t, t') = G_k^>(t', t) \quad (24)$$

The effective equations of motion to one loop that determine the time evolution of the scalar field are

$$\frac{d^2\phi(t)}{dt^2} + m^2(t)\phi(t) + \frac{\lambda}{6}\phi^3(t) + \frac{\lambda}{2}\phi(t) \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{U}_k^+(t)\mathcal{U}_k^-(t)}{2\omega_<(k)} \coth\left[\frac{\beta_i\omega_<(k)}{2}\right] = 0 \quad (25)$$

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + M^2(t)\right]\mathcal{U}_k^\pm = 0 \quad (26)$$

with (20) , (21).

This set of equations is too complicated to attempt an analytic solution. They will be dealt with numerically. However, before doing this, we should note that there are several features of this set of equations that reveal the basic physical aspects of the dynamics of the scalar field.

i): The effective evolution equations are **real**. The mode functions $\mathcal{U}_k^\pm(t)$ are complex conjugates of each other as may be seen from the time reversal symmetry of the equations, and the boundary conditions (21). This situation must be contrasted with the expression for the effective potential for the *analytically continued modes*.

ii): Consider the situation in which the initial configuration of the classical field is near the origin $\phi_i \approx 0$, for $t > 0$. The modes for which $\vec{k}^2 < (k_{max})^2$; $(k_{max})^2 = \mu^2 - \frac{\lambda}{2}\phi_i^2$ are *unstable*.

In particular, for early times ($t > 0$), when $\phi_i \approx 0$, these unstable modes behave approximately as

$$\mathcal{U}_k^+(t) = A_k e^{W_k t} + B_k e^{-W_k t} \quad (27)$$

$$\mathcal{U}_k^-(t) = (\mathcal{U}_k^+(t))^* \quad (28)$$

$$A_k = \frac{1}{2} \left[1 - i \frac{\omega_<(k)}{W_k} \right] ; B_k = A_k^* \quad (29)$$

$$W_k = \left[\mu^2 - \frac{\lambda}{2}\phi_i^2 - \vec{k}^2 \right]^{\frac{1}{2}} \quad (30)$$

Then the early time behavior of $(-iG_k(t, t))$ is given by

$$(-iG_k(t, t)) \approx \frac{1}{2\omega_<(k)} \left[1 + \frac{\mu^2 + m^2}{\mu^2 - \frac{\lambda}{2}\phi_i^2 - k^2} [\cosh(2W_k t) - 1] \right] \coth[\beta_i\omega_<(k)/2] \quad (31)$$

This early time behavior coincides with the Green's function of Guth and Pi[15] and Weinberg and Wu[16] for the inverted harmonic oscillators when our initial state (density matrix) is taken into account.

Our evolution equations, however, permit us to go beyond the early time behavior and to incorporate the non-linearities that will eventually shut off the instabilities.

These early-stage instabilities and subsequent growth of fluctuations and correlations, are the hallmark of the process of phase separation, and precisely the instabilities that trigger the phase transition.

It is clear from the above equations of evolution, that the description in terms of inverted oscillators will only be valid at very early times. At such times, the *stable* modes for which $\vec{k}^2 > (k_{max})^2$ are obtained from (27) , (28) , (29) by the analytic continuation $W_k \rightarrow -i\omega_>(k) = [\vec{k}^2 - \mu^2 + \frac{\lambda}{2}\phi_i^2]^{\frac{1}{2}}$.

For $t < 0$, $\mathcal{U}_k^+(t)\mathcal{U}_k^-(t) = 1$ and one obtains the usual result for the evolution equation

$$\frac{d^2\phi(t)}{dt^2} + \frac{dV_{eff}(\phi)}{d\phi} = 0$$

with $V_{eff}(\phi)$ the finite temperature effective potential but for $t < 0$ there are no unstable modes.

It becomes clear, however, that for $t > 0$ there are no *static* solutions to the evolution equations for $\phi(t) \neq 0$.

iii)Coarsening: as the classical expectation value $\phi(t)$ “rolls down” the potential hill, $\phi^2(t)$ increases and $(k_{max}(t))^2 = \mu^2 - \frac{\lambda}{2}\phi^2(t)$ decreases, and only the very long-wavelength modes remain unstable, until for a particular time t_s ; $(k_{max}(t_s))^2 = 0$. This occurs when $\phi^2(t_s) = 2\mu^2/\lambda$, which is the inflexion point of the tree level potential. In statistical mechanics this point is known as the “classical spinodal point” and t_s as the “spinodal time”[21, 22]. When the classical field reaches the spinodal point, all instabilities shut- off. From this point on, the dynamics is oscillatory and this period is identified with the “reheating” stage in cosmological scenarios[11, 12].

It is clear from the above equations of evolution, that the description in terms of inverted oscillators will only be valid at small positive times, as eventually the unstable growth will shut-off.

The value of the spinodal time depends on the initial conditions of $\phi(t)$. If the initial value ϕ_i is very near the classical spinodal point, t_s will be relatively small and there will not be enough time for the unstable modes to grow too much. In this case, the one-loop corrections for small coupling constant will remain perturbatively small. On the other hand, however, if $\phi_i \approx 0$, and the initial velocity is small, it will take a very long time to reach the classical spinodal point. In this case the unstable modes **may grow dramatically making the one-loop corrections non-negligible even for small coupling**. These initial conditions of small initial field and velocity are precisely the “slow rollover” conditions that are of interest in cosmological scenarios of “new inflation”.

The renormalization aspects have been studied in reference[18] and we refer the reader to that article for details.

3 Analysis of the Evolution

As mentioned previously within the context of coarsening, when the initial value of the scalar field $\phi_i \approx 0$, and the initial temporal derivative is small, the scalar field slowly rolls down the potential hill. But during the time while the scalar field remains smaller than the “spinodal” value, the unstable modes grow and the one-loop contribution grows as a consequence. For a “slow rollover” condition, the field remains very small ($\phi^2(t) \ll 2\mu^2/\lambda$) for a long time, and during this time the unstable modes grow exponentially. After renormalization, the stable modes give an oscillatory contribution which is bound in time, and for weak coupling remains perturbatively small at all times.

Then for a “slow rollover” situation and for weak coupling, only the unstable modes will yield to an important contribution to the one-loop correction. Thus, in the evolution equation for the scalar field, we will keep only the integral over the *unstable modes* in the one loop correction.

Phenomenologically the coupling constant in these models is bound by the spectrum of density fluctuations to be within the range $\lambda_R \approx 10^{-12} - 10^{-14}$ [11]. The stable modes will always give a perturbative contribution, whereas the unstable modes grow exponentially in time thus raising the possibility of giving a non-negligible contribution.

With the purpose of numerical analysis of the effective equations of motion, it proves convenient to introduce the following dimensionless variables

$$\tau = \mu_R t ; q = k/\mu_R \quad (32)$$

$$\eta^2(t) = \frac{\lambda_R}{6\mu_R^2} \phi^2(t) ; L^2 = \frac{m_R^2 + \frac{1}{2}\lambda_R \phi_i^2}{\mu_R^2} \quad (33)$$

To account for the change from the initial temperature to the final temperature ($T_i > T_c ; T_f < T_c$) we parametrize[35]

$$m^2 = \mu_R(0) \left[\frac{T_i^2}{T_c^2} - 1 \right] \quad (34)$$

$$\mu_R = \mu_R(0) \left[1 - \frac{T_f^2}{T_c^2} \right] \quad (35)$$

where the subscripts (R) stand for renormalized quantities, and $-\mu_R(0)$ is the renormalized zero temperature “negative mass squared” and $T_c^2 = 24\mu_R^2(0)/\lambda_R$. Furthermore, because $(k_{max}(t))^2 \leq \mu_R^2$ and $T_i > T_c$, for the unstable modes $T_i \gg (k_{max}(t))^2$ and we can take the high temperature limit $\coth[\beta_i \omega_<(k)/2] \approx 2T_i/\omega_<(k)$. Finally the effective equations of evolution for $t > 0$, become, after using $\omega_<^2 = \mu_R^2(q^2 + L^2)$ and keeping only the unstable modes as explained above ($q^2 < (q_{max}(\tau))^2$),

$$\frac{d^2}{d\tau^2} \eta(\tau) - \eta(\tau) + \eta^3(\tau) +$$

$$g\eta(\tau) \int_0^{(q_{max}(\tau))} q^2 \frac{\mathcal{U}_q^+(\tau)\mathcal{U}_q^-(\tau)}{q^2 + L^2} dq = 0 \quad (36)$$

$$\left[\frac{d^2}{d\tau^2} + q^2 - (q_{max}(t))^2 \right] \mathcal{U}_q^\pm(\tau) = 0 \quad (37)$$

$$(q_{max}(\tau))^2 = 1 - 3\eta^2(\tau) \quad (38)$$

$$g = \frac{\sqrt{6\lambda_R}}{2\pi^2} \frac{T_i}{T_c \left[1 - \frac{T_f^2}{T_c^2} \right]} \quad (39)$$

For $T_i \geq T_c$ and $T_f \ll T_c$ the coupling (39) is bound within the range $g \approx 10^{-7} - 10^{-8}$. The dependence of the coupling with the temperature reflects the fact that at higher temperatures the fluctuations are enhanced.

From (36) we see that the quantum corrections act as a *positive dynamical renormalization* of the “negative mass” term that drives the rolling down dynamics. It is then clear, that the quantum corrections tend to *slow down the evolution*.

In particular, if the initial value $\eta(0)$ is very small, the unstable modes grow for a long time before $\eta(\tau)$ reaches the spinodal point $\eta(\tau_s) = 1/\sqrt{3}$ at which point the instabilities shut off. If this is the case, the quantum corrections introduce substantial modifications to the classical equations of motion, thus becoming non-perturbative. If $\eta(0)$ is closer to the classical spinodal point, the unstable modes do not have time to grow dramatically and the quantum corrections are perturbatively small.

Thus we conclude that the initial conditions on the field determine whether or not the quantum corrections are perturbatively small.

Figures (1,2) depict the solutions for the classical (solid lines) and quantum (dashed lines) evolution.

For the numerical integration we have chosen $L^2 = 1$, the results are only weakly dependent on L , and taken $g = 10^{-7}$, we have varied the initial condition $\eta(0)$ but used $\frac{d\eta(\tau)}{d\tau}|_{\tau=0} = 0$.

We recall, from a previous discussion that $\eta(\tau)$ should be identified with the average of the field within a domain. We are considering the situation in which this average is very small, according to the usual slow-rollover hypothesis, and for which the instabilities are stronger.

In figure (1) we plot η vs τ for $g = 10^{-7}$, $\eta(0) = 2.3 \times 10^{-5}$, $\eta'(0) = 0$; $L = 1$. The solid line is the classical evolution, the dashed line is the evolution from the one-loop corrected equation of motion. We begin to see that the quantum corrections become important at $t \approx 10/\mu_R$ and slow down the dynamics. By the time that the *classical* evolution reaches the minimum of the classical potential at $\eta = 1$, the quantum evolution has just reached the classical spinodal point $\eta = 1/\sqrt{3}$. The quantum correction becomes large enough to change the sign of the “mass term” [18], the field continues its evolution towards larger values, however, because the velocity is different from zero. As η gets closer to the classical spinodal point, the instability shuts off and the quantum correction arising from the unstable modes become small. From the spinodal point onwards, the field evolves towards the minimum and

begins to oscillate around it. The quantum correction will be perturbatively small, as all the instabilities had shut-off. Higher order corrections, will introduce a damping term as quanta may decay into elementary excitations of the true vacuum.

Figure (2) shows a dramatic behavior for $\eta(0) = 2.258 \times 10^{-5}$; $\frac{d\eta(0)}{d\tau} = 0$ for the same values of the parameters as for figure (1). The unstable modes have enough time to grow so dramatically that the quantum correction becomes extremely large overwhelming the “negative mass” term near the origin. The dynamical time dependent potential, now becomes a *minimum* at the origin and the quantum evolution begins to *oscillate* near $\eta = 0$. The contribution of the unstable modes has become *non-perturbative*, and certainly our one- loop approximation breaks down.

As the initial value of the field gets closer to zero, the unstable modes grow for a very long time. At this point, we realize, however, that this picture cannot be complete. To see this more clearly, consider the case in which the initial state or density matrix corresponds exactly to the symmetric case. $\eta = 0$ is necessarily, by symmetry, a fixed point of the equations of motion. Beginning from the symmetric state, the field will *always remain* at the origin and though there will be strong quantum and thermal fluctuations, these are symmetric and will sample field configurations with opposite values of the field with equal probability.

In this situation, and according to the picture presented above, one would then expect that the unstable modes will grow indefinitely because the scalar field does not roll down and will never reach the classical spinodal point thus shutting-off the instabilities. What is missing in this picture and the resulting equations of motion is a self-consistent treatment of the unstable fluctuations, which must necessarily go beyond one- loop approximation.

4 Domain formation and growth:

The instabilities correspond to the growth of long-wavelength fluctuations and the formation of domains within which the field is correlated. These domains will grow as long as the instabilities persist. In order to understand the process of domain growth when the average value of the field remains zero it proves illuminating to understand the tree level correlations.

The relevant quantity of interest is the *equal time* correlation function

$$S(\vec{r}; t) = \langle \Phi(\vec{r}, t) \Phi(\vec{0}, t) \rangle \quad (40)$$

$$S(\vec{r}; t) = \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}} S(\vec{k}; t) \quad (41)$$

$$S(\vec{k}; t) = \langle \Phi_{\vec{k}}(t) \Phi_{-\vec{k}}(t) \rangle = (-i G_{\vec{k}}^{++}(t; t)) \quad (42)$$

where we have performed the Fourier transform in the spatial coordinates (there still is spatial translational and rotational invariance). Notice that at equal times, all the Green’s functions are equal, and we may compute any of them.

Clearly in an equilibrium situation this equal time correlation function will be time independent, and will only measure the *static correlations*. In the present case, however,

there is a non trivial time evolution arising from the departure from equilibrium of the initial state.

The unstable contribution to the Green function at equal times is given by (31), in this case for $\phi_i = 0$.

It is convenient to introduce the following dimensionless quantities

$$\kappa = \frac{k}{m_f} ; \quad L^2 = \frac{m_i^2}{m_f^2} = \frac{[T_i^2 - T_c^2]}{[T_c^2 - T_f^2]} ; \quad \tau = m_f t ; \quad \vec{x} = m_f \vec{r} \quad (43)$$

Furthermore for the unstable modes $\vec{k}^2 < m_f^2$, and for initial temperatures larger than the critical temperature $T_c^2 = 24\mu^2/\lambda$, we can approximate $\coth[\beta_i \omega_<(k)/2] \approx 2T_i/\omega_<(k)$. Then, at tree-level, the contribution of the unstable modes to the subtracted structure factor (42) $S^{(0)}(k, t) - S^{(0)}(k, 0) = (1/m_f)\mathcal{S}^{(0)}(\kappa, \tau)$ becomes

$$\mathcal{S}^{(0)}(\kappa, \tau) = \left(\frac{24}{\lambda[1 - \frac{T_f^2}{T_c^2}]} \right)^{\frac{1}{2}} \left(\frac{T_i}{T_c} \right) \frac{1}{2\omega_\kappa^2} \left(1 + \frac{\omega_\kappa^2}{W_\kappa^2} \right) [\cosh(2W_\kappa \tau) - 1] \quad (44)$$

$$\omega_\kappa^2 = \kappa^2 + L^2 \quad (45)$$

$$W_\kappa = 1 - \kappa^2 \quad (46)$$

To obtain a better idea of the growth of correlations, it is convenient to introduce the scaled correlation function

$$\mathcal{D}(x, \tau) = \frac{\lambda}{6m_f^2} \int_0^{m_f} \frac{k^2 dk}{2\pi^2} \frac{\sin(kr)}{(kr)} [S(k, t) - S(k, 0)] \quad (47)$$

The reason for this is that the minimum of the tree level potential occurs at $\lambda\Phi^2/6m_f^2 = 1$, and the inflexion (spinodal) point, at $\lambda\Phi^2/2m_f^2 = 1$, so that $\mathcal{D}(0, \tau)$ measures the excursion of the fluctuations to the spinodal point and beyond as the correlations grow in time.

At large τ (large times), the product $\kappa^2 \mathcal{S}(\kappa, \tau)$ in (47) has a very sharp peak at $\kappa_s = 1/\sqrt{\tau}$. In the region $x < \sqrt{\tau}$ the integral may be done by the saddle point approximation and we obtain for $T_f/T_c \approx 0$ the large time behavior

$$\mathcal{D}(x, \tau) \approx \mathcal{D}(0, \tau) \exp\left[-\frac{x^2}{8\tau}\right] \frac{\sin(x/\sqrt{\tau})}{(x/\sqrt{\tau})} \quad (48)$$

$$\mathcal{D}(0, \tau) \approx \left(\frac{\lambda}{12\pi^3} \right)^{\frac{1}{2}} \left(\frac{(\frac{T_i}{2T_c})^3}{[\frac{T_i^2}{T_c^2} - 1]} \right) \frac{\exp[2\tau]}{\tau^{\frac{3}{2}}} \quad (49)$$

Restoring dimensions, and recalling that the zero temperature correlation length is $\xi(0) = 1/\sqrt{2}\mu$, we find that for $T_f \approx 0$ the amplitude of the fluctuation inside a “domain” $\langle \Phi^2(t) \rangle$,

and the “size” of a domain $\xi_D(t)$ grow as

$$\langle \Phi^2(t) \rangle \approx \frac{\exp[\sqrt{2}t/\xi(0)]}{(\sqrt{2}t/\xi(0))^{\frac{3}{2}}} \quad (50)$$

$$\xi_D(t) \approx (8\sqrt{2})^{\frac{1}{2}}\xi(0)\sqrt{\frac{t}{\xi(0)}} \quad (51)$$

An important time scale corresponds to the time τ_s at which the fluctuations of the field sample beyond the spinodal point. Roughly speaking when this happens, the instabilities should shut-off as the mean square root fluctuation of the field $\sqrt{\langle \Phi^2(t) \rangle}$ is now probing the stable region. This will be seen explicitly below when we study the evolution non-perturbatively in the Hartree approximation and the fluctuations are incorporated self-consistently in the evolution equations. In zeroth order we estimate this time from the condition $3\mathcal{D}(0, t) = 1$, we use $\lambda = 10^{-12}$; $T_i/T_c = 2$, as representative parameters (this value of the initial temperature does not have any particular physical meaning and was chosen only as representative). We find

$$\tau_s \approx 10.15 \quad (52)$$

or in units of the zero temperature correlation length

$$t \approx 14.2\xi(0) \quad (53)$$

for other values of the parameter τ_s is found from the above condition on (49).

Clearly any perturbative expansion will fail because the propagators will contain unstable wavelengths, and the one loop term will grow faster than the zero order, etc.[17]. We now turn to a non-perturbative analysis (for details see[17]).

As the correlations and fluctuations grow, field configurations start sampling the stable region beyond the spinodal point. This will result in a slow down in the growth of correlations, and eventually the unstable growth will shut-off. When this happens, the state may be described by correlated domains with equal probability for both phases inside the domains. The expectation value of the field in this configuration will be zero, but inside each domain, the field will acquire a value very close to the value in equilibrium at the minimum of the *effective potential*. The size of the domain in this picture will depend on the time during which correlations had grown enough so that fluctuations start sampling beyond the spinodal point.

Since this physical picture may not be studied within perturbation theory, we now introduce a *non-perturbative* method based on a self-consistent Hartree approximation, which is implemented as follows: in the initial Lagrangian write

$$\frac{\lambda}{4!}\Phi^4(\vec{r}, t) = \frac{\lambda}{4}\langle \Phi^2(\vec{r}, t) \rangle \Phi^2(\vec{r}, t) + \left(\frac{\lambda}{4!}\Phi^4(\vec{r}, t) - \frac{\lambda}{4}\langle \Phi^2(\vec{r}, t) \rangle \Phi^2(\vec{r}, t) \right) \quad (54)$$

the first term is absorbed in a shift of the mass term

$$m^2(t) \rightarrow m^2(t) + \frac{\lambda}{2} \langle \Phi^2(t) \rangle$$

(where we used spatial translational invariance). The second term in (54) is taken as an interaction with the term $\langle \Phi^2(t) \rangle \Phi^2(\vec{r}, t)$ as a “mass counterterm”. The Hartree approximation consists of requiring that the one loop correction to the two point Green’s functions must be cancelled by the “mass counterterm”. This leads to the self consistent set of equations

$$\langle \Phi^2(t) \rangle = \int \frac{d^3k}{(2\pi)^3} (-iG_k^<(t, t)) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_<(k)} \mathcal{U}_k^+(t) \mathcal{U}_k^-(t) \coth[\beta_i \omega_<(k)/2] \quad (55)$$

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + m^2(t) + \frac{\lambda}{2} \langle \Phi^2(t) \rangle \right] \mathcal{U}_k^\pm = 0 \quad (56)$$

Before proceeding any further, we must address the fact that the composite operator $\langle \Phi^2(\vec{r}, t) \rangle$ needs one subtraction and multiplicative renormalization. As usual the subtraction is absorbed in a renormalization of the bare mass, and the multiplicative renormalization into a renormalization of the coupling constant.

At this stage our justification for using this approximation is based on the fact that it provides a non-perturbative framework to sum an infinite series of Feynman diagrams of the cactus type[8, 36].

It is clear that for $t < 0$ there is a self-consistent solution to the Hartree equations with equation (55) and

$$\begin{aligned} \langle \Phi^2(t) \rangle &= \langle \Phi^2(0^-) \rangle \\ \mathcal{U}_k^\pm &= \exp[\mp i\omega_<(k)] \end{aligned} \quad (57)$$

$$\omega_<^2(k) = \vec{k}^2 + m_i^2 + \frac{\lambda}{2} + \langle \Phi^2(0^-) \rangle = \vec{k}^2 + m_{i,R}^2 \quad (58)$$

where the composite operator has been absorbed in a renormalization of the initial mass, which is now parametrized as $m_{i,R}^2 = \mu_R^2[(T_i^2/T_c^2) - 1]$. For $t > 0$ we subtract the composite operator at $t = 0$ absorbing the subtraction into a renormalization of m_f^2 which we now parametrize as $m_{f,R}^2 = \mu_R^2[1 - (T_f^2/T_c^2)]$. We should point out that this choice of parametrization only represents a choice of the bare parameters, which can always be chosen to satisfy this condition. The logarithmic multiplicative divergence of the composite operator will be absorbed in a coupling constant renormalization consistent with the Hartree approximation[36, 37], however, for the purpose of understanding the dynamics of growth of instabilities associated with the long-wavelength fluctuations, we will not need to specify this procedure. After this subtraction procedure, the Hartree equations read

$$[\langle \Phi^2(t) \rangle - \langle \Phi^2(0) \rangle] = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_<(k)} [\mathcal{U}_k^+(t) \mathcal{U}_k^-(t) - 1] \coth[\beta_i \omega_<(k)/2] \quad (59)$$

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + m_R^2(t) + \frac{\lambda_R}{2} (\langle \Phi^2(t) \rangle - \langle \Phi^2(0) \rangle) \right] \mathcal{U}_k^\pm(t) = 0 \quad (60)$$

$$m_R^2(t) = \mu_R^2 \left[\frac{T_i^2}{T_c^2} - 1 \right] \Theta(-t) - \mu_R^2 \left[1 - \frac{T_f^2}{T_c^2} \right] \Theta(t) \quad (61)$$

with $T_i > T_c$ and $T_f \ll T_c$. With the self-consistent solution and boundary condition for $t < 0$

$$[\langle \Phi^2(t < 0) \rangle - \langle \Phi^2(0) \rangle] = 0 \quad (62)$$

$$\mathcal{U}_k^\pm(t < 0) = \exp[\mp i\omega_<(k)t] \quad (63)$$

$$\omega_<(k) = \sqrt{\vec{k}^2 + m_{iR}^2} \quad (64)$$

This set of Hartree equations is extremely complicated to be solved exactly. However it has the correct physics in it. Consider the equations for $t > 0$, at very early times, when (the renormalized) $\langle \Phi^2(t) \rangle - \langle \Phi^2(0) \rangle \approx 0$ the mode functions are the same as in the zeroth order approximation, and the unstable modes grow exponentially. By computing the expression (59) self-consistently with these zero-order unstable modes, we see that the fluctuation operator begins to grow exponentially.

As $(\langle \Phi^2(t) \rangle - \langle \Phi^2(0) \rangle)$ grows larger, its contribution to the Hartree equation tends to balance the negative mass term, thus weakening the instabilities, so that only longer wavelengths can become unstable. Even for very weak coupling constants, the exponentially growing modes make the Hartree term in the equation of motion for the mode functions become large and compensate for the negative mass term. Thus when

$$\frac{\lambda_R}{2m_{f,R}^2} (\langle \Phi^2(t) \rangle - \langle \Phi^2(0) \rangle) \approx 1$$

the instabilities shut-off, this equality determines the “spinodal time” t_s . The modes will still continue to grow further after this point because the time derivatives are fairly (exponentially) large, but eventually the growth will slow-down when fluctuations sample deep inside the stable region.

After the subtraction, and multiplicative renormalization (absorbed in a coupling constant renormalization), the composite operator is finite. The stable mode functions will make a *perturbative* contribution to the fluctuation which will be always bounded in time. The most important contribution will be that of the *unstable modes*. These will grow exponentially at early times and their effect will dominate the dynamics of growth and formation of correlated domains. The full set of Hartree equations is extremely difficult to solve, even numerically, so we will restrict ourselves to account *only* for the unstable modes. From the above discussion it should be clear that these are the only relevant modes for the dynamics of formation and growth of domains, whereas the stable modes, will always contribute perturbatively for weak coupling after renormalization.

Introducing the dimensionless ratios (43) in terms of $m_{f,R}$; $m_{i,R}$, (all momenta are now expressed in units of $m_{f,R}$), dividing (60) by $m_{f,R}^2$, using the high temperature approximation

$\coth[\beta_i \omega_<(k)/2] \approx 2T_i/\omega_<(k)$ for the unstable modes, and expressing the critical temperature as $T_c^2 = 24\mu_R/\lambda_R$, the set of Hartree equations (59, 60) become the following integro-differential equation for the mode functions for $t > 0$

$$\left[\frac{d^2}{d\tau^2} + q^2 - 1 + g \int_0^1 dp \left\{ \frac{p^2}{p^2 + L_R^2} [\mathcal{U}_p^+(t) \mathcal{U}_p^-(t) - 1] \right\} \right] \mathcal{U}_q^\pm(t) = 0 \quad (65)$$

with

$$\mathcal{U}_q^\pm(t < 0) = \exp[\mp i \omega_<(q) t] \quad (66)$$

$$\omega_<(q) = \sqrt{q^2 + L_R^2} \quad (67)$$

$$L_R^2 = \frac{m_{i,R}^2}{m_{f,R}^2} = \frac{[T_i^2 - T_c^2]}{[T_c^2 - T_f^2]} \quad (68)$$

$$g = \frac{\sqrt{24\lambda_R}}{4\pi^2} \frac{T_i}{[T_c^2 - T_f^2]^{\frac{1}{2}}} \quad (69)$$

The effective coupling (69) reflects the enhancement of quantum fluctuations by high temperature effects; for $T_f/T_c \approx 0$, and for couplings as weak as $\lambda_R \approx 10^{-12}$, $g \approx 10^{-7}(T_i/T_c)$.

The equations (65) may now be integrated numerically for the mode functions; once we find these, we can then compute the contribution of the unstable modes to the subtracted correlation function equivalent to (47)

$$\mathcal{D}^{(HF)}(x, \tau) = \frac{\lambda_R}{6m_{f,R}^2} [\langle \Phi(\vec{r}, t) \Phi(\vec{0}, t) \rangle - \langle \Phi(\vec{r}, 0) \Phi(\vec{0}, 0) \rangle] \quad (70)$$

$$3\mathcal{D}^{(HF)}(x, \tau) = g \int_0^1 dp \left(\frac{p^2}{p^2 + L_R^2} \right) \frac{\sin(px)}{(px)} [\mathcal{U}_p^+(t) \mathcal{U}_p^-(t) - 1] \quad (71)$$

In figure (3) we show

$$\frac{\lambda_R}{2m_{f,R}^2} (\langle \Phi^2(\tau) \rangle - \langle \Phi^2(0) \rangle) = 3(\mathcal{D}^{HF}(0, \tau) - \mathcal{D}^{HF}(0, 0))$$

(solid line) and also for comparison, its zeroth-order counterpart $3(\mathcal{D}^{(0)}(0, \tau) - \mathcal{D}^{(0)}(0, 0))$ (dashed line) for $\lambda_R = 10^{-12}$, $T_i/T_c = 2$. (this value of the initial temperature does not have any particular physical significance and was chosen as a representative). We clearly see what we expected; whereas the zeroth order correlation grows indefinitely, the Hartree correlation function is bounded in time and oscillatory. At $\tau \approx 10.52$, $3(\mathcal{D}^{(HF)}(0, \tau) - \mathcal{D}^{(HF)}(0, 0)) = 1$, fluctuations are sampling field configurations near the classical spinodal, fluctuations continue to grow, however, because the derivatives are still fairly large. However, after this time, the modes begin to probe the stable region in which there is no exponential growth. At this point $\frac{\lambda_R}{2m_{f,R}^2} (\langle \Phi^2(\tau) \rangle - \langle \Phi^2(0) \rangle)$, becomes small again because of the small coupling $g \approx 10^{-7}$, and the correction term becomes small. When it becomes smaller than one, the

instabilities set in again, modes begin to grow and the process repeats. This gives rise to an oscillatory behavior around $\frac{\lambda_R}{2m_{f,R}^2}(\langle\Phi^2(\tau)\rangle - \Phi^2(0)) = 1$ as shown in figure (3). In figure (4), we show the structure factors as a function of x for $\tau = 10$, both for zero-order (tree level) $\mathcal{D}^{(0)}$ (dashed lines) and Hartree $\mathcal{D}^{(HF)}$ (solid lines) for the same value of the parameters as for figure (3). These correlation functions clearly show the growth in amplitude and that the size of the region in which the fields are correlated increases with time. Clearly this region may be interpreted as a “domain”, inside which the fields have strong correlations, and outside which the fields are uncorrelated.

We see that up to the spinodal time $\tau_s \approx 10.52$ at which $\frac{\lambda_R}{2m_{f,R}^2}(\langle\Phi^2(\tau_s)\rangle - \Phi^2(0)) = 1$, the zeroth order correlation $3\mathcal{D}^{(0)}(0, \tau)$ is very close to the Hartree result. In fact at τ_s , the difference is less than 15%. For these values of the coupling and initial temperature, the zeroth order correlation function leads to $t_s \approx 10.15$, and we may use the zeroth order correlations to provide an analytic estimate for t_s , as well as the form of the correlation functions and the size of the domains. The fact that the zeroth-order correlation remains very close to the Hartree-corrected correlations up to times comparable to the spinodal is a consequence of the very small coupling. The stronger coupling makes the growth of domains much faster and the departure from tree-level correlations more dramatic[17]. For strong couplings domains will form very rapidly and only grow to sizes of the order of the zero temperature correlation length. The phase transition will occur very rapidly, and our initial assumption of a rapid supercooling will be unjustified. This situation for strong couplings, of domains forming very rapidly to sizes of the order of the zero temperature correlation length is the picture presented by Mazenko and collaborators[14]. However, for very weak couplings (consistent with the bounds from density fluctuations), our results indicate that the phase transition will proceed very slowly, domains will grow for a long time and become fairly large, with a typical size several times the zero temperature correlation length. In a sense, this is a self consistent check of our initial assumptions on a rapid supercooling in the case of weak couplings.

As we argued above, for very weak coupling we may use the tree level result to give an approximate bound to the correlation functions up to times close to the spinodal time using the result given by equation (49), for $T_f \approx 0$. Thus, we conclude that for large times, and very weakly coupled theories ($\lambda_R \leq 10^{-12}$) and for initial temperatures of the order of the critical temperature, the size of the domains $\xi_D(t)$ will grow typically in time as

$$\xi_D(t) \approx (8\sqrt{2})^{\frac{1}{2}} \xi(0) \sqrt{\frac{t}{\xi(0)}} \quad (72)$$

with $\xi(0)$ the zero temperature correlation length. The maximum size of a domain is approximately determined by the time at which fluctuations begin probing the stable region, this is the spinodal time t_s and the maximum size of the domains is approximately $\xi_D(t_s)$.

An estimate for the spinodal time, is obtained from equation (49) by the condition

$3\mathcal{D}(\tau_s) = 1$. For weakly coupled theories and $T_f \approx 0$, we obtain

$$\tau_s = \frac{t_s}{\sqrt{2}\xi(0)} \approx -\ln \left[\left(\frac{3\lambda}{4\pi^3} \right)^{\frac{1}{2}} \left(\frac{(\frac{T_i}{2T_c})^3}{[\frac{T_i^2}{T_c^2} - 1]} \right) \right] \quad (73)$$

5 FRW Cosmologies:

We now consider the case of scalar field in an homogeneous, isotropic and spatially flat FRW cosmology described by the metric

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 \quad (74)$$

The action and Lagrangian density are given by

$$S = \int d^4x \mathcal{L} \quad (75)$$

$$\mathcal{L} = a^3(t) \left[\frac{1}{2} \dot{\Phi}^2(\vec{x}, t) - \frac{1}{2} \frac{(\vec{\nabla} \Phi(\vec{x}, t))^2}{a(t)^2} - V(\Phi(\vec{x}, t)) \right] \quad (76)$$

$$V(\Phi) = \frac{1}{2} [m^2 + \xi \mathcal{R}] \Phi^2(\vec{x}, t) + \frac{\lambda}{4!} \Phi^4(\vec{x}, t) \quad (77)$$

$$\mathcal{R} = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \quad (78)$$

with \mathcal{R} the Ricci scalar. We have introduced the coupling to the Ricci scalar as it will be induced by renormalization (see below).

We can either follow the same method outlined above in terms of a path integral in the complex time plane, or alternatively one could evolve the density matrix in time by solving explicitly the Liouville equation, these two methods are equivalent.

We have followed[38] the second alternative in one-loop and Hartree approximation (thus providing another independent manner of tackling the problem).

The initial condition is assumed to correspond to a local equilibrium thermal ensemble of the adiabatic modes at an early time $t_o \rightarrow -\infty$ at a temperature $T_o = 1/\beta_o$. In the case of a de Sitter background, this initial condition corresponds to a thermal distribution of the Bunch-Davies modes, and for $T_o \rightarrow 0$ the density matrix describes a pure state, the Bunch-Davies vacuum.

Any description of quantum statistical mechanics needs to specify an initial density matrix whose subsequent time evolution is determined by the Hamiltonian. The assumption of local thermal equilibrium at an early time is somewhat arbitrary. Whether or not it corresponds to a physically realistic situation can only be answered within the realm of a theory that incorporates gravity, particle physics and statistical mechanics as dynamical ingredients. The choice of an initial condition for a density matrix will pervade any out of equilibrium dynamical calculation. If the system reaches thermal equilibrium after a long

time, the details on the initial conditions may ultimately be irrelevant, but this requires a deeper understanding.

Our goal is to obtain the evolution equation for the order parameter, as well as the fluctuations. The order parameter is the expectation value (in the time dependent density matrix) of the volume average of the scalar field

$$\phi(t) = \frac{1}{\Omega} \int d^3x \langle \Phi(\vec{x}, t) \rangle = \frac{1}{\Omega} \int d^3x \text{Tr} \hat{\rho}(t) \Phi(\vec{x}) = \frac{1}{\Omega} \int d^3x \text{Tr} \hat{\rho}(t_o) U^{-1}(t, t_o) \Phi(\vec{x}, t_o) U(t, t_o) \quad (79)$$

where Ω is the comoving volume, and the scale factors cancel between the numerator (in the integral) and the denominator. $U(t, t_o)$ is the time evolution operator with boundary condition $U(t_o, t_o) = 1$, t_o is the time at which the initial ensemble is specified.

To one-loop order we find the effective evolution equations for the order parameter[38]

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) + \lambda\phi \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3} \frac{[\mathcal{U}_{1k}^2(t) + \mathcal{U}_{2k}^2(t)]}{2a^3(t)W_k(t_o)} \coth[\beta_o \hbar W_k(t_o)/2] = 0 \quad (80)$$

where the mode functions $\mathcal{U}_{\alpha,k}(t)$; $\alpha = 1, 2$ are real and satisfy

$$\ddot{\mathcal{U}}_{\alpha k} - \frac{3}{2} \left(\frac{\ddot{a}}{a} + \frac{1}{2} \frac{\dot{a}^2}{a^2} \right) \mathcal{U}_{\alpha k} + \left(\frac{\vec{k}^2}{a^2(t)} + V''(\phi_{cl}(t)) \right) \mathcal{U}_{\alpha k} = 0 \quad (81)$$

$$\mathcal{U}_{1k}(t_o) = 1 \quad ; \quad \mathcal{U}_{2k}(t_o) = 0 \quad (82)$$

$$\dot{\mathcal{U}}_{1k}(t_o) = \frac{3}{2} \frac{\dot{a}(t_o)}{a(t_o)} \quad ; \quad \dot{\mathcal{U}}_{2k}(t_o) = W_k(t_o) = \left[\frac{\vec{k}^2}{a^2(t_o)} + V''(\phi_{cl}(t_o)) \right]^{\frac{1}{2}} \quad (83)$$

where $\phi_{cl}(t)$ is the solution to the *classical* equations of motion.

We can see clearly that the time dependence of the mode functions is very important. If the scale factor varies rapidly with time as for example in a radiation dominated or de Sitter epoch, at no time will the notion of an effective potential be valid for the time evolution of the order parameter.

The method lends itself to a self-consistent non-perturbative Hartree approximation. Invoking a Hartree factorization as in a previous section, we also derived the equations of motion. Introducing the fluctuation $\eta(\vec{x}, t) = \Phi(\vec{x}, t) - \phi(t)$ and the Hartree “frequencies”

$$\mathcal{V}^{(2)}(\phi) = V''(\phi) + \frac{\lambda}{2} \langle \eta^2 \rangle \quad (84)$$

we find

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) + \lambda\phi \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3} \frac{[\mathcal{U}_{1k}^2(t) + \mathcal{U}_{2k}^2(t)]}{2a^3(t)\mathcal{W}_k(t_o)} \coth[\beta_o \hbar \mathcal{W}_k(t_o)/2] = 0 \quad (85)$$

$$\left[\frac{d^2}{dt^2} - \frac{3}{2} \left(\frac{\ddot{a}}{a} + \frac{1}{2} \frac{\dot{a}^2}{a^2} \right) + \frac{\vec{k}^2}{a^2(t)} + V''(\phi(t)) + \right.$$

$$\lambda \frac{\hbar}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{[\mathcal{U}_{1k}^2(t) + \mathcal{U}_{2k}^2(t)]}{2a^3(t)\mathcal{W}_k(t_o)} \coth [\beta_o \hbar \mathcal{W}_k(t_o)/2] \mathcal{U}_{\alpha k} = 0 \quad (86)$$

$$\mathcal{U}_{1k}(t_o) = 1 \quad ; \quad \mathcal{U}_{2k}(t_o) = 0 \quad (87)$$

$$\dot{\mathcal{U}}_{1k}(t_o) = \frac{3\dot{a}(t_o)}{2a(t_o)} \quad ; \quad \dot{\mathcal{U}}_{2k}(t_o) = \mathcal{W}_k(t_o) = \left[\frac{\vec{k}^2}{a^2(t_o)} + \mathcal{V}^{(2)}(\phi(t_o)) \right]^{\frac{1}{2}} \quad (88)$$

The one-loop and Hartree equations present new features not present in our previous analysis in Minkowski space: first the “friction” term arising from the expansion, and secondly the fact that physical wave-vectors are red-shifted. The red-shift will tend to enhance the instabilities as more wavelengths become unstable, but the presence of the horizon will ultimately constrain the final size of the correlated regions.

We are currently studying the numerical evolutions of these equations and expect to report soon on details of the dynamics of the phase transition in these cosmologies[38].

6 Renormalization Aspects

To understand the renormalization procedure, it proves convenient to introduce the complex combination of mode functions

$$\mathcal{U}_k^\pm(t) = \mathcal{U}_{1k} \mp i\mathcal{U}_{2k} \quad (89)$$

$$\mathcal{U}_k^\pm(t_o) = 1 \quad (90)$$

$$\dot{\mathcal{U}}_k^\pm(t) = \frac{3\dot{a}(t_o)}{2a(t_o)} \mp i\mathcal{W}_k(t_o) \quad (91)$$

in the case of the one-loop approximation $\mathcal{W}_k(\phi)$ must be replaced by $W_k(\phi)$. We will now analyze the renormalization aspects for the Hartree approximation, the one-loop case may be obtained easily from this more general case. We need to understand the divergences in the integral

$$I = \int \frac{d^3 k}{(2\pi)^3} \frac{\mathcal{U}_k^+(t)\mathcal{U}_k^-(t)}{2a^3(t)\mathcal{W}_k(t_o)} \quad (92)$$

The divergences in this integral will be determined from the large-k behavior of the mode functions that are solutions to the differential equation (86) with the boundary conditions (90, 91). The large-k behavior of these functions may be obtained in a WKB approximation by introducing the WKB function

$$\mathcal{D}_k(t) = \exp \left[\int_{t_o}^t R(t') dt' \right] \quad (93)$$

$$\mathcal{D}_k(t_o) = 1 \quad (94)$$

satisfying the differential equation

$$\left[\frac{d^2}{dt^2} - \frac{3}{2} \left(\frac{\ddot{a}}{a} + \frac{1}{2} \frac{\dot{a}^2}{a^2} \right) + \frac{\vec{k}^2}{a^2(t)} + V''(\phi(t)) + \frac{\lambda}{2} \langle \eta^2(\vec{x}, t) \rangle \right] \mathcal{D}_k = 0 \quad (95)$$

with $\langle \eta^2(\vec{x}, t) \rangle$ being the self-consistent integral in the Hartree equation (86) (the one-loop approximation is obtained by setting this term to zero in the above equation). The mode functions of interest \mathcal{U}^{pm} are obtained as linear combinations of the WKB function and its complex conjugate. The coefficients to be determined from the boundary condition at t_o . The function $R(t)$ obeys a Riccati (WKB) equation

$$\dot{R} + R^2 - \frac{3}{2} \left(\frac{\ddot{a}}{a} + \frac{1}{2} \frac{\dot{a}^2}{a^2} \right) + \frac{\vec{k}^2}{a^2(t)} + V''(\phi(t)) + \frac{\lambda}{2} \langle \eta^2(\vec{x}, t) \rangle = 0 \quad (96)$$

We propose a solution to this equation of the form

$$R = \frac{-ik}{a(t)} + R_o(t) - \frac{iR_1(t)}{k} + \frac{R_2(t)}{k^2} + \dots \quad (97)$$

and find the time dependent coefficient by comparing powers of k . We find

$$R_o(t) = \frac{\dot{a}(t)}{2a(t)} \quad (98)$$

$$R_1(t) = \frac{a(t)}{2} \left[-\frac{\mathcal{R}}{6} + V''(\phi(t)) + \frac{\lambda}{2} \langle \eta^2(\vec{x}, t) \rangle \right] \quad (99)$$

$$R_2(t) = -\frac{1}{2} \frac{d}{dt} [a(t)R_1(t)] \quad (100)$$

Finally we obtain

$$I = \frac{1}{8\pi^2} \frac{\Lambda^2}{a^2(t)} + \frac{1}{8\pi^2} \ln \left(\frac{\Lambda}{K} \right) \left[\frac{\dot{a}^2(t_o)}{a^2(t)} - \left(-\frac{\mathcal{R}}{6} + V''(\phi(t)) + \frac{\lambda}{2} \langle \eta^2(\vec{x}, t) \rangle \right) \right] + \text{finite} \quad (101)$$

where we have introduced a renormalization point K , and the finite part depends on time, temperature and K . For renormalization in the one-loop approximation, $\langle \eta^2 \rangle$ does not appear in the logarithmic divergent term in (101) (as it does not appear in the differential equation for the mode functions up to one loop). There are several physically important features of the divergent structure obtained above. The first term (quadratically divergent) reflects the fact that the physical momentum cut-off is being red-shifted by the expansion. This term will not appear in dimensional regularization.

Secondly, the logarithmic divergence contains a term that reflects the initial condition (the derivative of the expansion factor at the initial time t_o). The initial condition breaks any remnant symmetry (for example in de Sitter space there is still invariance under the de Sitter group, but this is also broken by the initial condition at an arbitrary time t_o). Thus this term is not forbidden, and its appearance does not come as a surprise.

The renormalization conditions in the Hartree approximation are obtained by requiring that the equation for the mode functions be finite[38]. Thus we obtain

$$m_B^2 + \frac{\lambda_B \hbar}{16\pi^2} \frac{\Lambda^2}{a^2(t)} + \frac{\lambda_B \hbar}{16\pi^2} \ln \left(\frac{\Lambda}{K} \right) \frac{\dot{a}^2(t_o)}{a^2(t)} = m_R^2 \left[1 + \frac{\lambda_B \hbar}{16\pi^2} \ln \left(\frac{\Lambda}{K} \right) \right] \quad (102)$$

$$\lambda_B = \lambda_R \left[1 + \frac{\lambda_B \hbar}{16\pi^2} \ln \left(\frac{\Lambda}{K} \right) \right] \quad (103)$$

$$\xi_B = \xi_R + \frac{\lambda_B \hbar}{16\pi^2} \ln \left(\frac{\Lambda}{K} \right) \left(\xi_R - \frac{1}{6} \right) \quad (104)$$

Notice that the conformal coupling $\xi = 1/6$ is a *fixed point* under renormalization.

7 High Temperature Limit:

One of the payoffs of understanding the large- k behavior of the mode functions (as obtained in the previous section via the WKB method) is that it permits a straightforward evaluation of the high temperature limit. The finite temperature contribution is determined by the integral

$$I_{\beta_o} = \int \frac{d^3k}{(2\pi)^3} \frac{[\mathcal{U}_{1k}^2(t) + \mathcal{U}_{2k}^2(t)]}{a^3(t)W_k(t_o)} \frac{1}{e^{\beta_o \hbar \mathcal{W}_k(t_o)} - 1} \quad (105)$$

For large temperature, only momenta $k \geq T_o$ contribute. Thus the leading contribution is determined by the first term of the function $R(t)$ of the previous section. After some straightforward algebra we find

$$I_{\beta_o} = \frac{1}{12} \left[\frac{k_B T_o a(t_o)}{\hbar a(t)} \right]^2 + \mathcal{O}(1/T_o) + \dots \quad (106)$$

Thus we see that the leading high temperature behavior reflects the physical red-shift, in the cosmological background and it results in an effective time dependent temperature

$$T_{eff}(t) = T_o \left[\frac{a(t_o)}{a(t)} \right]$$

To this leading order, the expression obtained for the time dependent effective temperature corresponds to what would be obtained for an *adiabatic* (isentropic) expansion for blackbody-type radiation (massless relativistic particles) in the cosmological background.

This behavior, however, only appears at *leading* order in the high temperature expansion. Higher order terms do not seem to have this form[38] as both the coupling to the curvature and the mass term distort the spectrum from the blackbody form.

8 Conclusions:

We provided a dynamical picture of the time evolution of a weakly coupled inflaton scalar field theory undergoing a typical second order phase transition in Minkowski space-time. The shortcomings of the usual approach based on the equilibrium effective potential and the necessity for a description out of equilibrium were pointed out. The non-equilibrium evolution equations for the order parameter were derived to one-loop and integrated for

“slow-rollover” initial conditions. We pointed out that the instabilities responsible for the onset of the phase transition and the process of domain formation, that is the growth of long-wavelength fluctuations, introduced dramatic corrections to the “slow-rollover” picture. The net effect of these instabilities is to *slow even further* the evolution of the order parameter (average of the scalar field) and for “slow rollover” initial conditions (the scalar field very near the false vacuum initially) the dynamics becomes *non-perturbative*.

We introduced a self-consistent approach to study the dynamics out of equilibrium in this situation. It is found that for very weakly coupled theories domains growth to large sizes typically several times the zero temperature correlation length and that the growth obeys a scaling law at intermediate times.

In strongly coupled theories the phase transition will occur much faster and domains will only grow to sizes of the order of the zero temperature correlation length.

The one-loop equations of evolution for the scalar order parameter were obtained in spatially flat FRW cosmologies. These equations reveal that at no times is the approximation of an effective potential valid. We have also obtained the evolution equations in a non-perturbative self-consistent Hartree approximation. These equations present new features that illuminate the fact that the process of phase separation in cosmological settings is more subtle.

The physical wave vectors undergo a red-shift and more wavelengths become unstable, thus enhancing the instability. On the other hand the presence of the horizon and the “friction” term in the equations of motion, will prevent domains from growing bigger than the horizon size. Eventually there is a competition of time scales that must be understood more deeply to obtain any meaningful conclusion about formation and growth of domains in FRW cosmologies. Work on these issues is in progress[38].

The renormalization aspects were studied and we point out that in a time dependent background, divergences appear that depend on the initial conditions. We also studied the leading behavior in the high temperature limit showing to this order that the temperature is red-shifted as in adiabatic expansion.

This formalism could also be applied to chaotic initial conditions[11]

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